

Heegaard Splittings and Bounds for Closed Hyperbolic 3-Manifolds

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Abstract

We will develop a way to Delaunay Triangulate \( [LL] \) closed and compact hyperbolic 3-manifolds, and use this method of triangulation to produce Heegaard splittings of such 3-manifolds. This method of triangulating and splitting manifolds will allow us to produce a universal constant \( K = \frac{\text{vol}(B_{\epsilon/4})}{\text{vol}(B_{\epsilon/4})^2} \), where \( \epsilon \) is half a Margulis constant for \( H^3 \), such that the Heegaard genus of a closed hyperbolic 3-manifold \( M \) is at most \( K \text{vol}(M) \).

1 Introduction

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1.1 Topology

For basic 3-manifold topology, which I will assume in this paper, the reader should see Allen Hatcher’s survey Notes on Basic 3-Manifold Topology, [Hat], and Jesse Johnson’s notes, [Joh]. However, we will review a few items which are extremely important for the topics in this paper.

A handlebody is 3-manifold with boundary homeomorphic to the closure of a regular neighborhood of a graph in \( \mathbb{R}^3 \). Equivalently, a handlebody is a manifold obtained by glueing 1-handles along disjoint disks in the boundary of a 3-ball. A handlebody \( H \) obtained by glueing on \( g \) 1-handles has boundary the closed surface of genus \( g \), and we call \( H \) a genus \( g \) handlebody. Given a closed 3-manifold \( M \), a Heegaard splitting for \( M \) is a triple \(( F, H_1, H_2) \) where \( H_1 \) and \( H_2 \) are handlebodies of the same genus \( g \) and \( F \) is a properly embedded closed genus \( g \) surface in \( M \), and \( M = H_1 \cup_F H_2 \). The Heegaard genus of \( M \) is the smallest number \( g(M) \) such that there exists a Heegaard splitting \(( F, H_1, H_2) \) of \( M \) with the genus of \( F \) equal to \( g(M) \).

Sometimes we will use this following third characterization of a handlebody: let \( B_1, \ldots, B_n \) be a collection of 3-balls, and let \( D_1, \ldots, D_m, D'_1, \ldots, D'_m \) be a collection of pairwise disjoint disks embedded in the union of the boundaries of the \( B_i \). For each pair \( D_i \) and \( D'_i \) let \( f_i : D_i \to D'_i \) be a homeomorphism, and let \( H \) denote the manifold with boundary obtained by gluing along the homeomorphisms \( f_i \). If \( H \) is connected, then \( H \) is a handlebody of genus \( m - n + 1 \). If the reader is not immediately convinced that this formulation is equivalent to the definitions above, see [Joh] for detailed proofs.

Example 1. Consider the case \( M = S^3 \) and \( F = S^2 \). Since by Alexander’s Theorem every 2-sphere in \( S^3 \) bounds a ball and the complement of a ball in \( S^3 \) is also a ball, splitting \( M \) along \( F \) yields \( M = B_1 \cup_F B_2 \), where \( B_1 \) and \( B_2 \) are 3-balls. This is a genus 0 Heegaard splitting of \( S^3 \), so the Heegaard genus of \( S^3 \) is 0.

The obvious question here is this: does every closed 3-manifold have such a splitting? Luckily, the best possible existence result holds.

Proposition 1. Every closed 3-manifold admits a Heegaard splitting.
Proof. Let $T$ be some fixed triangulation of a 3-manifold $M$. By Moise [Moi], such a triangulation exists. Take a closed regular neighborhood $H_1$ of the 1-skeleton of $T$ which is, by definition, a handlebody. Let $F$ be a tetrahedron in $T$. Then the complement of $F \cap \text{int}(H_1)$ in $F$ is a ball, and if $F'$ is a tetrahedron adjacent to $F$, then the balls $F \cap \text{int}(H_1)$ and $F' \cap \text{int}(H_1)$ formed this way intersect in a disk in the face $F \cap F'$ which they share. Hence the complement of the interior of $H_1$ in $M$ is a connected collection of balls glued together along disks, and thus is a handlebody $H_2$. If $S$ is the surface which is the common boundary of both $H_1$ and $H_2$, then by construction $H_1 \cup_S H_2$ is a Heegaard splitting for $M$.

What is less clear is how one might define a Heegaard Splitting of a compact 3-manifold with non-empty boundary. One way to do this is to introduce the notion of a compression body, which is the analogue of a handlebody in the non-empty boundary case. We will define a compression body as a compact, orientable, irreducible 3-manifold $H$ which has a boundary component $\partial H_+$, called the exterior boundary of $H$, such that the induced map $\pi_1(\partial H_+) \to \pi_1(H)$ is a surjection. We denote the remaining boundary components by $\partial H_-$. We define the genus of $H$ to be the genus of $\partial H_+$.

We will redefine a handlebody simply as a compression body $H$ with $\partial H_- = \emptyset$, and leave it to the reader to check that the definition here is equivalent to the one given in the first section.

A Heegaard Splitting of genus $g$ of a compact 3-manifold, then, is a triple $(F,H_1,H_2)$ where $F$ is a closed surface of genus $g$ embedded in $M$ which splits $M$ into two disjoint embedded compression bodies $H_1$ and $H_2$ such that $\partial(H_1) = \partial(H_2) = F$. Furthermore

**Proposition 2.** Every compact 3-manifold $M$ admits a Heegaard Splitting.

A proof of the above proposition can be found in [Sch], and is very similar in nature to the proof that every closed 3-manifold admits a Heegaard splitting.

As above, we define the Heegaard genus of a compact $M$ to be the smallest such number $g(M)$ such that $M$ admits a Heegaard splitting of genus $g(M)$. For a considerably more amount of information about Heegaard splittings of compact 3-manifolds, see [Sch].

### 1.2 Hyperbolic Manifolds

Now, we will consider some basic hyperbolic geometry. We will state much of the theory only for 3-manifolds, since this is the case we are interested in. Let $M$ be a closed Riemannian 3-manifold. We say that $M$ is hyperbolic if $M$ admits a metric with constant sectional curvature equal to $-1$. It is well-known that this is equivalent to the existence of a discrete, torsion-free subgroup $\Gamma \subseteq \text{Isom}(\mathbb{H}^3)$ such that $M$ is isometric to $\mathbb{H}^3/\Gamma$. A closed 3-manifold $M$ is said to be atoroidal if $\pi_1(M)$ does not contain any subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and is called irreducible if every 2-sphere in $M$ bounds a ball. Amazingly we have the following theorem, conjectured by Thurston and due to Perelman:

**Theorem.** A closed orientable 3-manifold admits if hyperbolic metric if and only if it is irreducible, atoroidal, and has infinite fundamental group.

Let $M$ be a hyperbolic 3-manifold. For a point $x$ in $M$, define the injectivity radius of $M$ at $x$, denoted $\text{inj}_x(M)$ to be the radius of the largest isometrically embedded hyperbolic ball centered at $x$. Equivalently, $\text{inj}_x(M)$ is half the length of the shortest non-trivial simple closed geodesic curve passing through $x$. Define the injectivity radius of $M$, $\text{inj}(M)$ to be the infimum over all $x$ in $M$ of $\text{inj}_x(M)$.

Interestingly, the following result of Margulis will allow us to describe exactly what the parts of $M$ where the injectivity radius is “small” looks like.

**Proposition 3** (Margulis Lemma). For all $n$ in $\mathbb{N}$, there exists an $\epsilon_n > 0$ such that for any subgroup $\Gamma \subseteq \text{Isom}(\mathbb{H}^3)$ which acts properly discontinuously and for any $x$ in $\mathbb{H}^3$ the group $\Gamma_{\epsilon_n}(x) = \langle \gamma \in \Gamma : d(x, \gamma(x)) \leq \epsilon_n \rangle$ is almost nilpotent.

We recall what it means for a group $G$ to be nilpotent: define the $k$th commutator $G^k$ of $G$ recursively as $G^k = [G, G^{k-1}]$ with $G^1 = [G, G]$. The group $G$ is nilpotent if $G^k$ is trivial for some $k$ in $\mathbb{N}$. Furthermore, $G$ is said to be almost nilpotent (abelian) if $G$ has a finite-index subgroup which is nilpotent (abelian).
For each \( n \), any positive number less than the number \( \epsilon_n \) will be called a Margulis constant for \( H^n \).

For the rest of the paper let \( \mu \leq \epsilon_3 \) be a Margulis constant for \( H^3 \). For a hyperbolic 3-manifold \( M \), define the thick-part of \( M \), denoted \( M_{\text{thick}} \), to be the submanifold of \( M \) where the injectivity radius is greater than or equal to \( \mu \). Define the thin-part of \( M \), denoted \( M_{\text{thin}} \), to be the closure of the complement of \( M_{\text{thick}} \). We have the following result.

**Theorem (Thick-Thin Decomposition).** Let \( M \) be a hyperbolic 3-manifold. Then the components of \( M_{\text{thin}} \) are homeomorphic to either

1. A solid torus \( D^2 \times S^1 \),
2. \( T \times [0, \infty) \), where \( T \) is a torus,
3. or \( S^1 \times \mathbb{R} \times [0, \infty) \).

Furthermore, if \( M \) is compact then \( M_{\text{thin}} \) consists of only solid tori.

A proof for the above theorem can be found in many books in hyperbolic manifolds; see for example [BP92].

However, the basic idea of the proof is that, by the Margulis Lemma, the group \( \Gamma_{\mu}(x) \subseteq \Gamma \) consists of either only hyperbolic elements (elements with exactly two fixed points on the sphere at infinity) which share an axis or of only parabolic elements (elements with exactly one fixed point on the sphere at infinity) which share a fixed point; if \( \Gamma_{\mu}(x) \) consists of hyperbolic elements, it acts by translation and rotation on a collar neighborhood of a geodesic in \( H^3 \), and so produces a solid torus. If the group \( \Gamma_{\mu}(x) \) consists of parabolics, then it acts by isometries on a horoball. Since horospheres inherit a Euclidean metric from the ambient hyperbolic one, \( \Gamma_{\mu}(x) \) must either be isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) or \( \mathbb{Z} \), and hence the resulting quotient of a horoball by one of these groups is either homeomorphic to the product of a torus with \([0, \infty)\) or \( S^1 \times \mathbb{R} \times [0, \infty) \). The last statement of the theorem is obvious.

### 1.3 Rank versus Genus

The rank of a manifold \( M \) (or the rank of \( \pi_1(M) \), we will say both) is the minimal number of generators required in a presentation of \( \pi_1(M) \). Since for a Heegaard splitting of \( M \) by compression bodies \( H_1 \) and \( H_2 \) the fundamental group \( \pi_1(H_i) \) surjects onto \( \pi_1(M) \), one sees immediately that if \( M = H_1 \cup_S H_2 \) is a genus \( g \) Heegaard splitting of \( M \) then the following inequality holds

\[
\text{rank } \pi_1(M) \leq g(M).
\]

Furthermore, one might ask when equality holds. There is an example of a Seifert manifold in which the rank is strictly less than the Heegaard genus which can be found in [BH84], but we do have the following conjecture

**Conjecture.** Let \( M \) be a closed hyperbolic 3-manifold. Then the Heegaard genus of \( M \) is equal to the rank of \( \pi_1(M) \).

If this conjecture turns out not to be true, then one could ask if we can bound the difference \( g(M) - \text{rank}(\pi_1(M)) \). Also, one might consider which (if any) classes of 3-manifolds have Heegaard genus some fixed number \( k \) greater than their rank. The above conjecture if obviously a special case of this question for \( k = 0 \). For a rather more comprehensive overview of the above problem, see the paper [Sou] by Juan Souto.

### 2 Delaunay Triangulations

In [LL] the reader will find conditions for constructing so-called Delaunay Triangulations of hyperbolic (actually, Riemannian) manifolds. The basic idea is that given a discrete set of points \( S \) in a hyperbolic manifold \( M \) satisfying certain density requirements and being sufficiently generic, one can form a geodesic triangulation of \( M \) in which four points of \( S \) form the vertices of a tetrahedron if they lie on the boundary of an embedded metric ball which contains no other points of \( S \). However, [LL] is a survey, so we will develop here, in detail, these results in the setting of hyperbolic 3-manifolds.
2.1 Triangulations of Closed 3-Manifolds

We say that a hyperbolic 3-manifold $M = \mathbb{H}^3 / \Gamma$ is $\epsilon$-thick for some positive number $\epsilon$ if the injectivity radius of $M$ is greater than $\epsilon$ at all points of $M$. Note that the thick-part of a manifold $M$ for some Margulis constant $\mu$ is an $\mu/2$-thick manifold, possibly with boundary. Let $M$ be a closed hyperbolic 3-manifold which is $\epsilon/2$-thick, and let $X$ be a set of points in $M$ maximal with respect to the property that for any two points $x$ and $y$ in $X$, we have $d(x, y) > \epsilon/2$ and such that no five points of $X$ lie on the boundary of an isometrically embedded hyperbolic ball. We say such a set $X$ is generically $\epsilon/2$-sparse. To form a triangulation of $M$ we proceed as follows: if four points of $X$ lie on the boundary of an isometrically embedded (closed) hyperbolic 3-ball $B$ such that $B$ contains no other points of $X$, then those four points form the vertices of a geodesic tetrahedron.

Of course, it remains to check that by such a process actually obtains a triangulation of $M$. The following results show this. Also, the reader should note that in what follows that we will assume $X$ contains at least 4 points. We begin with a quick proposition:

Lemma 1. Let $B$ be an embedded hyperbolic ball in $M$ containing only four points $a, b, c, d$ of $X$, all of which lie on $\partial B$. Then the radius of $B$ is less than $\epsilon/2$.

Proof. Suppose that the radius of $B$ was strictly greater than $\epsilon/2$. Then the center of $B$, say $p$, is a point of $M$ such that the distance from $p$ to $a, b, c, d$ is greater than $\epsilon/2$, and which is no closer to any other point of $X$. This contradicts the maximality of the set $X$. □

Lemma 2. No four points of $X$ which lie on the boundary of an embedded 3-ball containing no other points of $X$ are on the boundary of an hyperbolic 2-ball $H$ in an embedded hyperbolic plane in $M$.

Proof. Suppose four such points exist. Let $B$ be the smallest radius embedded hyperbolic 3-ball containing these points in its boundary, and say $B$ has radius $r$. Clearly, our four points lie on the equator of $B$, and so the center of $B$, say $p$ lies on 2-ball $H$ containing them. The first step of this proof contains a technique which we will use repeatedly, so I will state it detail.

Consider the family of 3-balls $\{B_t\}$ defined as follows: Let $\gamma$ be the geodesic perpendicular to $H$ at $p$. Since $H$ divides $B$ into two halves, choose one, say $B_+$, and let $\gamma'$ be the ray of $\gamma$ which extends on the side of $B_+$. Let $B_t$ be the ball centered at the point a distance $t$ from $p$ on $\gamma'$, with radius $R = \cosh^{-1}(\cosh(r) \cosh(t))$. The boundary of $H$ lies on $B_t$, since by basic hyperbolic trigonometry $\cosh(R) = \cosh(r) \cosh(t)$, and hence $B_t$ contains our four original points. Also, it is clear that the family $\{B_t\}$ varies continuously, since $R$ is a continuous function of $t$.

Then by compactness of $M$ there exists a smallest value $t$ for which $B_t$ contains another points of $X$ in $\partial B_t$. Hence $B_t$ is an embedded hyperbolic ball which contains five points of $X$ in its boundary, but this contradicts our choice of the set $X$. □

We will use this to prove the following:

Lemma 3. Let $a, b, c, d$ be four distinct points in $X$ which lie on the boundary of an embedded hyperbolic ball $R$ which contains no other points of $X$. Then $R$ is the unique isometrically embedded hyperbolic ball which satisfies this property.

Proof. Suppose there were another such hyperbolic ball, say $R'$. Let $A, B, C, D$ be lifts of $a, b, c, d$ under the covering map $\mathbb{H}^3 \to M$. We can certainly find lifts of $R$ and $R'$, say $Q$ and $Q'$, such that $Q$ and $Q'$ contain at least one of the points $A, B, C, D$ in common, say $A \in Q \cap Q'$. Suppose that $Q$ and $Q'$ both contain all the lifts. Then I claim that there is a unique hyperbolic ball passing through $A, B, C, D$, so that $Q = Q'$.

To show this claim we look in the Poincare Disc model, and note that hyperbolic spheres are just Euclidean spheres with (possibly) different centers. Indeed, spheres centered at the origin are certainly Euclidean balls. Since $\text{Isom}(\mathbb{H}^3)$ acts transitively and preserves spheres, the result follows for spheres centered at other points in $\mathbb{H}^3$. Then basic Euclidean analytic geometry, for example in [OG52], tells us that there is a unique sphere through any four points which do not lie in a plane. Therefore the same result holds in hyperbolic space.
Now assume that $Q$ contains a lift $B$ of $b$ and $Q'$ contains a distinct lift $B'$ of $b$. The fact that they both project to $b$ implies that there exists an isometry $\gamma$ in $\Gamma$ which moves $B$ to $B'$. Since the radii of $Q$ and $Q'$ are less than $\epsilon/2$, the distance from $Q$ to $Q'$ is less than $2\epsilon$, and hence when we project from $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$, there exists a closed geodesic loop of length less than $2\epsilon$ in $M$. This contradicts $M$ being $\epsilon$-thick. \qed

What remains to check is that picking a set of points $X$ and forming tetrahedra as above actually results in a triangulation of the manifold $M$. Suppose that we have four points, $a, b, c, d$ in $X$ which lie on the boundary of an embedded hyperbolic ball $B$. Suppose also that we have another point $e$ in $X$ which lies on the same side of the plane containing $a, b, c$, and such that $a, b, c, e$ lie on the boundary of an embedded hyperbolic ball which contains no other points of $X$ on its interior. Then the tetrahedra formed with vertices $a, b, c, d$ and $a, b, c, e$ intersect in their interiors. We would like to guarantee that the tetrahedra in our triangulation intersect only along faces. The next lemma guarantees this bad situation does not happen. The above lemmata imply the following desired result: let $X = \{x_1, \ldots, x_n\}$ be a set of points in $M$ maximal with respect to the condition that $d(x_i, x_j) \geq \epsilon/2$ for all $i \neq j$ and such that no five points in $X$ lie on the boundary of an embedded hyperbolic ball. $X$ is finite since $M_{\text{thick}}$ is compact. Pick three points $a, b, c$ of $X$ such that the minimal radius embedded hyperbolic ball whose boundary contains the three points and which contains no other points of $X$. Using a continuous family of isometrically embedded balls of increasing radius as in the proof of Lemma 5, there exists a fourth point $d$ such that there exists an embedded ball containing $a, b, c$ and $d$ in its boundary which contains no other points of $X$. Let $a, b, c, d$ form the vertices of a geodesic tetrahedron.

**Lemma 4.** Consider a set of points $a, b, c, d, e$ in $X$ such that $a, b, c, d$ lie on the boundary of an isometrically embedded 3-ball $B$ which contains no other points of $X$, and such that $e$ lies on the same side of the plane containing $a, b, c$ as the point $d$. Then an embedded hyperbolic ball containing $a, b, c, e$ must also contain $d$ in its interior.

**Proof.** With notation as above, let $r$ be the radius of the ball $B$. As in the proof of Lemma 2, consider the continuously varying family of embedded hyperbolic balls $\{B_t\}$ whose centers are moving perpendicular to the plane containing $a, b, c$, with $B_0 = B$ and which satisfy the following: since the plane containing $a, b, c$ divides $B$ into two connected subsets, we require that the centers of the $B_t$ lie in the subset containing $d$. Since $d$ lies on the same side of this plane as the center of $B$, the balls $B_t$ with $t > 0$ must contain $d$ in their interiors. But the point $e$ also lies on the same side of the plane, so there exists some $t > 0$ such that $e$ lies on the boundary of $B_t = B'$, which must contain $d$ since all of the $B_t$ contain $d$. \qed

The obvious corollary, and the reason the above proposition exists, is that distinct tetrahedra obtained via the process above only intersect along faces, and their interiors do not intersect at all. Furthermore:

**Lemma 5.** If $a, b, c, d$ in $X$ form the vertices of a tetrahedron formed by the process above, then there is another point $e$ in $X$, distinct from $a, b, c, d$ such that $a, b, c, e$ are contained in the boundary of an embedded hyperbolic ball $B$ which contains no other points of $X$ in its interior.

**Proof.** Let $r$ be the radius of the smallest embedded hyperbolic ball which contains $a, b, c$ in its boundary, and denote such a ball by $B$. Let $\{B_t\}$ be a continuously varying family of embedded hyperbolic balls as in Lemma 2, with $B_0 = B$ and with the property that for $s < t$ the center of $B_s$ is closer to the center the ball containing $a, b, c, d$ than the center of $B_t$ is. By compactness of $M$, there exists a smallest value for $t$ such that $B_t$ contains another point of $X$ in $\partial B_t$. Hence $B_t$ is an embedded hyperbolic ball containing $a, b, c, e$ in its boundary, and also not containing any other points of $X$. Since the balls $B_t$ are moving away from the ball containing $d$, we do not have $d = e$. \qed

The above lemmata imply the following desired result: let $X = \{x_1, \ldots, x_n\}$ be a set of points in $M$ maximal with respect to the condition that $d(x_i, x_j) \geq \epsilon/2$ for all $i \neq j$ and such that no five points in $X$ lie on the boundary of an embedded hyperbolic ball. $X$ is finite since $M_{\text{thick}}$ is compact. Pick three points $a, b, c$ of $X$ such that the minimal radius embedded hyperbolic ball whose boundary
contains the three points and which contains no other points of \( X \). Using a continuous family of isometrically embedded balls of increasing radius as in the proof of Lemma 5, there exists a fourth point \( d \) such that there exists an embedded ball containing \( a, b, c \) and \( d \) in its boundary which contains no other points of \( X \). Let \( a, b, c, d \) form the vertices of a geodesic tetrahedron. Lemma 5 allows one to continue this process to create more tetrahedra. Also

**Lemma 6.** Every point \( y \in M \) is contained in one of the tetrahedra obtained in the above manner, and hence we actually get a triangulation of \( M \).

**Proof.** We can assume \( d(y,x) < \epsilon/2 \) for some \( x \) in \( X \), otherwise \( y \) is at least \( \epsilon/2 \) from any point of \( X \), which contradicts the maximality of \( X \). Thus \( y \) is contained in an \( \epsilon/2 \)-ball around \( x \), say \( B_x \).

The above arguments imply that \( x \) is a vertex of at least one tetrahedron \( T_1 \). Then \( T_1 \cap B_x = C_1 \) is a cone, and say the edges of \( T_1 \) intersect \( \partial B_x \) at the points \( a, b, c \). If \( y \) is contained in \( C_1 \), then we are done. If not, construct another tetrahedron \( T_2 \) via Lemma 5, containing the points \( a, b, x \). If \( T_2 \cap B_x \) does not contain \( y \), then repeat the above process. Clearly each \( T_i \cap B_x \) has finite volume, and \( B_x \) does as well, so it stops after a finite number of steps, hence \( y \) must be contained in one of the \( T_i \cap B_x \), and so the result is proved.

Summarizing the above, we have proved the following:

**Theorem 1.** Given a closed hyperbolic 3-manifold \( M \) and a set of points \( X = \{x_1, \ldots, x_n\} \) generically \( \epsilon/2 \)-sparse, there exists a geodesic triangulation of \( M \) with the points of \( X \) as its vertices.

Such a triangulation of \( M \) will be called a Delaunay Triangulation.

### 2.2 Triangulations of Compact 3-Manifolds with \( \partial M \neq \emptyset \)

The above discussion is fine and dandy for closed hyperbolic manifolds, but what happens when \( M \) has boundary? The answer is exactly what one expects: in “most” places in \( M \), nothing, but one must be careful near the \( \partial M \).

We will be concerned in this paper only with hyperbolic 3-manifolds with boundary which are thick-parts of closed manifolds. Hence we will cheat and assume that any compact manifold \( M \) in this section is the thick-part of a closed manifold \( M' \).

Firstly, we need to know something about Delaunay Triangulations of closed hyperbolic 2-manifolds:

**Remark.** The results of the above Section 2.1 carry through exactly for surfaces. That is, if we pick a set of points \( X \) in a closed hyperbolic surface \( S \) maximal with respect to the condition that the distance between any two points of \( X \) is greater than \( \epsilon/2 \) and such that no four points lie on an embedded hyperbolic circle, then \( X \) forms the vertices of a triangulation constructed as follows: if \( a, b, c \) in \( X \) are three points which are contained on the boundary of an embedded hyperbolic 2-ball which contains no other points of \( X \), then \( a, b, c \) form the vertices of a geodesic triangle. Such a triangulation is called a Delaunay Triangulation of \( S \).

To form a Delaunay Triangulation of an \( \epsilon \)-thick compact hyperbolic manifold \( M \), we proceed as follows: let \( S \) denote the (possibly disconnected) boundary of \( M \). Choose a Delaunay Triangulation \( S \), whose set of vertices we will denote \( X_\partial \) for obvious reasons. Then, pick a set of points \( X \), such that \( X_\partial \subseteq X \) and such that \( X \) satisfies the properties in Section 2.1. Continue to form a triangulation as in Section 2.1, but consider embedded balls in the manifold \( M' \) rather than \( M \). This is because points near \( \partial M \) have embedded balls of radius \( \epsilon \) in \( M' \), but not in \( M \).

Using the same techniques as in the above section, it follows that Theorem 1 also holds for compact manifolds which are thick-parts of closed hyperbolic 3-manifolds. We have shown

**Theorem 2.** Let \( M \) be a compact hyperbolic 3-manifold homeomorphic to the thick part of a closed 3-manifold \( M \). If \( X_\partial \) is a set of points in \( \partial M \) which is generically \( \epsilon/2 \)-sparse, and if \( X \) is a generically \( \epsilon/2 \)-sparse set of points in \( M \) containing \( X_\partial \), then there exists a geodesic triangulation of \( M \) with the points of \( X \) as its vertices.
3 Bounds

There is a well-known result in the theory of hyperbolic 3-manifolds which states that for a finite-volume hyperbolic manifold $M$, the rank of $\pi_1(M)$ is bounded above by some constant $K$ times the volume of $M$. We will not give all the details of the proof here, but the basic idea is that one chooses a collection $X$ of points in $M_{\text{thick}}$ maximal with respect to the property that for each pair of points in the collection $x$ and $y$, the distance from $x$ to $y$ is greater than $\epsilon/2$. Then, one forms a graph $G$ in $M$ by connecting two points in $X$ if they are within $2\epsilon$ of one another. Then induced map $\pi_1(G) \to \pi_1(M_{\text{thick}})$ is a surjection, and obviously the map $\pi_1(M_{\text{thick}}) \to \pi_1(M)$ is also a surjection. Hence, the rank of $\pi_1(G)$ is greater than the rank of $\pi_1(M)$. Then, by how we picked the set $X$, each vertex of $G$ has a ball of radius $\epsilon/4$ which is disjoint from the balls around other vertices. Since one can bound the genus of the graph $G$ by a universal linear function of the number of vertices in $G$, we get a linear bound for the rank of $\pi_1(G)$ in terms of volume. Hence

$$\text{rank } \pi_1(M) \leq K \text{vol}(M)$$

as we had desired.

Since we are considering the relationship between the rank of the fundamental group of a hyperbolic 3-manifold and its Heegaard genus, one might ask if the same, or a similar, result holds for Heegaard genus. The answer is yes.

**Proposition 4.** There exists a universal constant $K$ such that for any closed hyperbolic 3-manifold $M$ the inequality $g(M) \leq K \text{vol}(M)$ holds.

**Proof.** Since $M$ is closed, $M_{\text{thin}}$ consists on solid tori. So $M_{\text{thick}}$ is a compact manifold with boundary consisting of disjoint tori. If we consider a Heegaard splitting $(F, H_1, H_2)$ of $M_{\text{thick}}$, the compression bodies $H_1$ and $H_2$ have $\partial(H_1)$- consisting of a finite number of tori.

Glueing the tori in $M_{\text{thin}}$ onto $M_{\text{thick}}$ results in turning these compression bodies to handlebodies $H'_1$ and $H'_2$, since the fundamental group of $\partial(H_1)$ surjects onto $\pi_1(H_1')$ and hence onto $\pi_1(H_2')$. Hence $(F, H'_1, H'_2)$ is a Heegaard splitting of $M$.

It follows that the Heegaard genus of $M$ is at most the Heegaard genus of $M_{\text{thick}}$. By Theorem 2, we can form a Delaunay Triangulation $\Sigma$ of $M$ using a generically $\epsilon/2$-sparse set $X \subseteq M$. Moreover, by Proposition 2, we can find a Heegaard splitting of $M_{\text{thick}}$ by thickening the 1-skeleton $\Sigma_1$ of our triangulation. Hence the Heegaard genus of $M_{\text{thick}}$ is less than the rank of $H_1(\Sigma_1)$, which is exactly $\{\text{number of edges in } \Sigma_1\} - \{\text{number of vertices}\} + 1$. Denote the number of edges by $e$ and the number of vertices by $v$. Clearly also, $e - v + 1 \leq e$, so thus far we have $g(M) \leq e$.

Furthermore, since $X$ is generically $\epsilon/2$-sparse, around each point is centered an embedded $\epsilon/4$-ball, denoted $B_{\epsilon/4}$, which does not intersect the corresponding ball about any other vertex. Hence

$$v \leq \frac{\text{vol}(M)}{\text{vol}(B_{\epsilon/4})}$$

Also, by Lemma 1, the maximal length of an edge in $\Sigma_1$ is $\epsilon$. Consider a vertex $p$ in $\Sigma_1$. Then any other vertex in $\Sigma_1$ connected to $p$ by an edge must be contained in the ball of radius $\epsilon$ centered at $p$. Since each of these vertices has a ball of radius $\epsilon/4$ centered at it, and disjoint from other such balls, the number of vertices connected to $p$ by an edge, and thus the number of edges containing $p$, denoted $e_p$, is bounded by $rac{\text{vol}(B_{\epsilon/4})}{\text{vol}(B_{\epsilon/4})}$.

Moreover, the total number of edges $e$ is bounded above by the number of edges around each point times the number of points, that is, $e_p v$. Thus

$$e \leq e_p v \leq \frac{\text{vol}(B_{\epsilon/4})}{\text{vol}(B_{\epsilon/4})} \frac{\text{vol}(M)}{\text{vol}(B_{\epsilon/4})}$$

and so

$$g(M) \leq \frac{\text{vol}(B_{\epsilon/4})}{\text{vol}(B_{\epsilon/4})} \frac{\text{vol}(M)}{\text{vol}(B_{\epsilon/4})}$$

as desired. \qed
From a paper [Mey], we get lower bound for a Margulis constant \(\mu \geq 0.104\). Using this value, we see that the constant \(K = 1.36 \times 10^7\), which is certainly absurdly large. One would expect a much better bound, since a Heegaard splitting obtained by the triangulation method of Proposition 2 is likely of much higher genus than a minimal splitting, and hence simply bounding the number of edges around each point leads to a poor bound.

Furthermore, one might ask whether the best relationship that can hold between Heegaard genus and volume is linear. It seems feasible that one might construct some example of a closed hyperbolic 3-manifold \(M\) which fibers over the circle \(S^1\), and contains some non-trivial geodesic loop \(\gamma\) which is homotopic into the fiber subgroup of \(\pi_1(M)\). Then by a result of Kerckoff noted in [Ago02], if one removes such a curve the remaining manifold \(M' = M - \gamma\) is still hyperbolic. Looking a \(k\)-fold covers of \(M'\) corresponding to the \(k\)-fold cyclic cover of \(S^1\), we see that the boundary of such a cover \(M'_k\) consists of \(k\) tori, and hence the Heegaard genus is greater than \(k\). Therefore

\[
\frac{g(M'_k)}{\text{vol}(M'_k)} \geq \frac{k}{k \text{vol}(M)} \geq \frac{1}{\text{vol}(M)} = L
\]

for all \(k\). So we have the following conjecture:

**Conjecture.** The best bound for Heegaard genus in terms of volume for a compact hyperbolic manifold is linear in the volume.

**Further reading and background**

For more information on things related to this subject, the interested reader should also see [Moi], [Kir96], [HS88], [Breb], [Brea], and [BCW04], as well as the aforementioned references.

**References**


[Sou] Juan Souto. Geometry, heegaard splittings and rank of the fundamental group of hyperbolic 3-manifolds.