

# Polynomial Interpolation with MATLAB

Everyone knows that given two points  $(x_1, y_1)$  and  $(x_2, y_2)$  there is a unique line  $L$  which passes through these points. In this lab we will investigate what happens if we increase the number of points, and instead of seeking a line, look for a polynomial curve to fit the data.

## I. Parabolas

What happens with three points?

**Exercise 1:** Given three points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ , with  $x_1$ ,  $x_2$  and  $x_3$  distinct, do you think there is a unique parabola  $f(x) = a_0 + a_1x + a_2x^2$  which passes through all three points? (Don't use MATLAB here.)

In order to deal with specific numbers, let's fix  $x_1 = 0$ ,  $x_2 = 1/2$  and  $x_3 = 1$ . Suppose we wanted to find a parabola  $f(x) = a_0 + a_1x + a_2x^2$  which passes through  $(0, 1)$ ,  $(1/2, -1)$  and  $(1, 2)$ .

**Exercise 2:** Write down (by hand) a linear system whose solution gives  $a_0, a_1, a_2$ . Enter the augmented matrix for this system into MATLAB and use the `rref` command to find the desired parabola. Then type the command

```
>> f = inline('a_0 + a_1*x + a_2*x^2')
```

where you've replaced  $a_0, a_1, a_2$  with the values you found for them. Finally call the command

```
>> ezplot(f, [-0.1, 1.1])
```

You should get a graph of your parabola.

Next we want to investigate what happens to our parabola if we use the same  $x$ -values  $(0, 1/2, \text{ and } 1)$ , but vary the  $y$ -values of our three points. To this end, convert the linear system you wrote down in Exercise 2 into one of the form

$$X \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where  $X$  is a certain  $3 \times 3$  matrix.

**Exercise 3:** Enter the matrix  $X$  into MATLAB. Call the command

```
>> Z = X^(-1)
```

to find the inverse of  $X$ . Use this to re-solve the problem in Exercise 2 by multiplying  $Z$  by the column vector  $[1; -1; 2]$ . Use the same method to find the equation of a parabola which passes through the points  $(0, 5)$ ,  $(\frac{1}{2}, 6)$  and  $(1, 1)$ . Make a plot of your parabola.

The matrix  $X$  is a special type known as a *Vandermonde matrix*. If we enter the command

```
>> V = vander([0; 1/2; 1])
```

into MATLAB we almost get  $X$ , except--what is the difference? We could overcome this by using the command

```
>> X = fliplr(V)
```

## II. Parabolas, again

Let's examine what happens in a bit greater generality. To this end, call the command

```
>> syms y1 y2 y3
```

this will tell MATLAB that  $y_1$ ,  $y_2$  and  $y_3$  are "symbolic variables" (not numbers). Now try

```
>> a = Z*[y1; y2; y3]
```

This should output a vector which shows how the coefficients of your parabola,  $a_0, a_1, a_2$ , depend on the  $y$ -coordinates of your points (remember, we are keeping the  $x$ -coordinates fixed).

**Exercise 4:** Suppose we want to know  $f(2)$ . Write an expression for  $f(2)$  which depends on  $y_1, y_2$  and  $y_3$ . Suppose we keep  $y_1$  and  $y_3$  fixed, but add one to  $y_2$ . How does  $f(2)$  change?

## III. Quintics!

**Exercise 5:** Find the equation of the quintic which passes through the six points  $(-1, 1)$ ,  $(0, 2)$ ,  $(1, 1)$ ,  $(4, 1)$ ,  $(5, -2)$ , and  $(6, 3)$ . You probably want to use the `vander` (& `fliplr`) command. Make a graph.

## IV. Vandermonde Determinant

Input the following commands

```
>> syms x1 x2 x3
>> X = [1 x1 x1^2; 1 x2 x2^2; 1 x3 x3^2]
>> det(X)
```

This will compute the determinant of a general 3x3 Vandermonde Matrix. It is a bit non-obvious what this result is so try

```
>> factor(ans)
```

**Exercise 6:** By modifying the above commands, create a general 4x4 Vandermonde matrix, and find and factor its determinant. By comparing this with the 3x3 (and even the 2x2) case, try to guess a general formula for the  $n \times n$  Vandermonde determinant.  
(For the brave: Try to prove this formula using a row-reduction argument.)  
Does this result mean we can always do polynomial interpolation?